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EDITORIAL

# Lanchester resurgent? The mathematics of terrorism risk

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## Abstract

**Purpose** – The purpose of this editorial is to consider whether or not the classical "Lanchester equations" of military combat are useful for modeling the financial risks associated with contemporary terrorist attacks.

**Design/methodology/approach** – The paper begins by describing Lanchester's original model and its realm of applicability; then identifies shortcomings of the original equations, which, having been aggravated by differences between classical military combat and modern terrorist engagements, impede the application of the Lanchester paradigm in today's world. Finally, the paper explores whether or not these obstacles can be overcome by appropriate extensions of Lanchester's mathematical theory.

**Findings** – The principal result is that the Lanchester equations may be extended in a very natural way to include stochastic elements, difficult-to-quantify components, and various force asymmetries, thereby enabling the modeling of engagements between conventional and terrorist forces. Specifically, a family of diffusion processes is proposed to capture the terrorists' progress toward destroying a target, and provide a method for explicitly calculating the probability of target destruction.

**Originality/value** – The editorial seeks to model a category of catastrophe risk – terrorist attacks – for which the current mathematical literature (both military and financial) is somewhat limited.

**Keywords** – Terrorism, Risk assessment, Forecasting, Diffusion, Mathematical modelling, Differential equations

**Paper type** Viewpoint

Following the terrorist attacks of September 11, 2001, the US Congress passed the Terrorism Risk Insurance Act (TRIA) of 2002 to "establish a temporary Federal program that provides for a transparent system of shared public and private compensation for insured losses resulting from acts of terrorism[1]." In return for requiring US property-liability insurers to include terrorism coverage in certain critical lines of business, the legislation supplemented private reinsurance coverage for terrorism-related losses through the end of 2005. Two subsequent extensions of TRIA[2] have carved out a far from "temporary" role for the US Federal Government in financing terrorism risk.

As observed in a previous editorial (Powers, 2005), a necessary condition for private insurers and reinsurers to remain in the terrorism-risk market is the industry's confidence that total losses can be forecast with sufficient accuracy. In the present editorial, I will consider the potential use of the classical "Lanchester equations" of military combat in forecasting the frequency of successful terrorist attacks.

The author is indebted to Bruce I. Gudmundsson, both for introducing him to the Lanchester equations and for explaining their historical context and applicability.

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### *Ad hoc* models

Writing in a prior issue of this journal, Major (2002) proposed that the conditional probability of destruction of a target $i$, given that target $i$ is selected for attack by terrorists, be expressed as:

$$p_M = \exp\left(-\frac{A_i D_i}{\sqrt{V_i}}\right)\left(\frac{A_i^2}{A_i^2 + V_i}\right), \qquad (1)$$

where $A_i$ denotes the size of the forces assigned by the terrorists to attack $i$, $D_i$ denotes the size of the forces assigned by government (and possibility private security) to defend $i$, and $V_i$ denotes the value of $i$ as a target (which is assumed to have a square-root relationship to the target's physical presence). In this formulation, the first factor on the right-hand side of equation (1) represents the probability that the terrorists avoid detection prior to their attack (derived from a simple search model), and the second factor represents the probability that the terrorists are then successful in destroying the target (derived from a dose-response model).

Powers and Shen (2006) replaced the above formula with:

$$p_{PS} = \exp\left(-\frac{A_i^s D_i^s}{V_i^s}\right)\left(\frac{A_i^c}{A_i^c + D_i^c}\right), \qquad (2)$$

where the parameters $s, c > 0$ allow the relative impacts of $A_i$, $D_i$, and $V_i$ to manifest decreasing ($s, c < 1$), constant ($s, c = 1$), or increasing ($s, c > 1$) returns to scale. The biggest conceptual difference between equations (1) and (2) is the substitution of a power of $D_i$ for a power of $V_i$ in the denominator of the second factor (representing the terrorists' probability of success in destroying the target once they have avoided detection). Essentially, Powers and Shen (2006) viewed the second factor as the probability that the terrorists prevail in their combat engagement with the forces defending the target, and used a heuristic argument based upon a classical gambler's ruin model to justify its functional form (which consequently is decreasing in $D_i$).

Somewhat curiously, both equations (1) and (2) possess the property that the right-hand side approaches zero as $A_i \to \infty$[3]. What this means is that as the terrorists' forces increase in magnitude, the disadvantage of size in terms of avoiding detection eventually outweighs the benefit of size in combat. While this implication may be realistic in certain scenarios, it is easily challenged. For example, the September 11 attacks suggest a small role for detection in even the boldest of attacks when the target is inadequately defended.

To provide a more rigorous framework for the study of terrorism combat – and in particular, to attempt to resolve the relative effects of physical detection and combat performance – we consider whether or not this form of contemporary conflict is amenable to the mathematical models of conventional military combat. Specifically, we explore whether the Lanchester equations of conventional combat may be extended appropriately to capture the various characteristics of modern terrorism.

### The Lanchester equations

By far the most widely studied mathematical model of military combat is that proposed by Lanchester (1916), which may be described by a system of differential equations of the form:

$$\frac{dA}{dt} = -k_1 A^{\alpha_1} D^{\delta_1}, \quad (3)$$

$$\frac{dD}{dt} = -k_2 A^{\alpha_2} D^{\delta_2}, \quad (4)$$

where $A = A(t)$ and $D = D(t)$ denote, respectively, the sizes of the attackers' and defenders' forces at time $t \geq 0$; $k_1$, $k_2$ are positive real-valued parameters denoting, respectively, the defenders' and attackers' effective destruction rates; and $\alpha_1$, $\alpha_2$ and $\delta_1$, $\delta_2$ are real-valued parameters reflecting the fundamental nature of the combat under study. In his original formulation, Lanchester (1916) considered two cases – one for "ancient" warfare, in which $\alpha_1 = 1,\ \ \delta_1 = 1,\ \ \alpha_2 = 1,\ \ \delta_2 = 1$, and one for "modern" warfare, in which $\alpha_1 = 0,\ \ \delta_1 = 1,\ \ \alpha_2 = 1,\ \ \delta_2 = 0$.

The rationale for the former model arises from hand-to-hand combat, in which the number of potential micro-engagements is given by the product of the two armies' forces, so that the rate of attrition of each side's forces at any time $t$ is proportional to this product. Solving the system of differential equations under this assumption yields the condition:

$$\frac{k_2}{k_1} > [<] \frac{D(0)}{A(0)}, \quad (5)$$

for the attackers (defenders) to win. The latter model is intended to reflect a type of combat in which the two armies fire upon each other from a distance, so that the rate of attrition of each side's forces at any time $t$ is simply proportional to the size of the opposing army's forces. Under this assumption, the system of differential equations yields the condition:

$$\frac{k_2}{k_1} > [<] \left[\frac{D(0)}{A(0)}\right]^2, \quad (6)$$

for the attackers (defenders) to win. The principal conclusion to be drawn from Lanchester's original analysis is that the ratio of the opposing armies' initial forces (i.e. $D(0)/A(0)$) plays a greater role in "modern" combat, where it is raised to the second power in condition (6), than in "ancient" combat, where it is raised to only the first power in condition (5).

Although the Lanchester equations have enjoyed some success in military applications – especially during World War II, when they were used extensively by the Allies to allocate reinforcements and logistical support – they possess several shortcomings of both theoretical and practical significance. These include:

- the assumption of homogeneous forces (i.e. both $A(t)$ and $D(t)$ change continuously over time, so that the loss of a tank cannot be distinguished from the loss of a soldier);
- the purely deterministic formulation (i.e. the army with greater forces is certain to win);
- the absence of certain difficult-to-quantify components (specifically, terrain, weather, and morale); and
- the failure to recognize certain asymmetries between armies (specifically, differences in objectives, information, and weaponry).

As military conflicts moved from the more conventional "modern" warfare of World Wars I and II to the more limited but protracted engagements of the Cold War and the War on Terror, the significance of the above drawbacks became more pronounced, and the Lanchester paradigm lost favor among military analysts. Despite a limited stream of continuing research, Lanchester theory largely has been replaced by simulation and role-playing techniques over the past few decades.

## Terrorism combat

So why attempt a Lanchester approach to terrorism – where the nature of the conflict is far from deterministic, where unusual terrain plays a major role, and where the asymmetries of objectives (instilling fear vs maintaining stability), information (surprise attacks vs constant vigilance), and weaponry (suicide bombers, airplanes, etc. vs a more conventional arsenal) are so extreme?

The answer is quite simple. The principal advantage offered by the Lanchester approach – which can never be achieved by simulation or role-playing – is analytical tractability. Thus, given that the motivation for our discussion is to compute the conditional probability of destruction of a target $i$, given that target $i$ selected for attack, the ability to compute this probability explicitly is worth an attempt at overcoming the many serious obstacles discussed above. Somewhat surprisingly, most of these shortcomings can be addressed in a reasonably straightforward manner.

We begin by replacing equations (3) and (4) by the system of stochastic differential equations[4]:

$$\mathrm{d}A = -\frac{k_1}{v^q} AD\mathrm{d}t + \sigma_1 \mathrm{d}Z_1, \tag{7}$$

$$\mathrm{d}D = -k_2 A\mathrm{d}t + \sigma_2 \mathrm{d}Z_2, \tag{8}$$

where $\mathrm{d}Z_1$, $\mathrm{d}Z_2$ are standard Brownian motions; $\sigma_1 = \sigma_1(A, D, t) > 0$, $\sigma_2 = \sigma_2(A, D, t) > 0$ are the associated infinitesimal standard deviations; $v$ denotes the three-dimensional volume of the target under attack; and $q$ denotes a power-transformation parameter used to recognize the appropriate domain of combat (e.g. $q = 1/3$ if a building can be attacked through only its ground-level perimeter, $q = 2/3$ if a building can be attacked anywhere along its surface, as by a fuel-filled airplane, and $q = 1$ if a bomb can be planted anywhere within a building).

Although any formulation with continuous values for $A$ and $D$ will model heterogeneous forces only approximately, the above system possesses several noteworthy improvements over the original Lanchester model. First, it introduces an explicit stochastic structure into an otherwise deterministic framework. Second, it captures the role of terrain to some extent through the parameter $q$, and is largely unaffected by changes in weather and morale since terrorist attacks are generally short in duration. Third, it captures the asymmetric information associated with a surprise attack on the target through the functional forms of the infinitesimal drifts on the right-hand sides of equations (7) and (8). This is because:

- the rate of attrition of the attackers' forces at any time $t$ is proportional to both the size of the defenders' forces – since all of the defenders' forces are available to fire on the enemy – as well as the size of the attackers' forces divided by $v^q$ – since the defenders do not know the attackers' actual physical locations within the combat domain; and

- the rate of attrition of the defenders' forces at any time $t$ is proportional to only the size of the attackers' forces – since the attackers know the defenders' physical locations.

Finally, it is able to recognize asymmetries in weaponry merely by appropriately adjusting the destruction-rate parameters $k_1$ and $k_2$ to have the correct units.

These enhancements leave one problem unaddressed – the failure to recognize asymmetries in objectives. However, this remaining issue is easily resolved in finding the probability of a successful terrorist attack, where one must specify a definition of "victory" for the attackers.

Essentially, we seek a mathematical expression for the probability that the process described by equations (7) and (8) reaches the attackers' "victory" state, $[A^{(A)}, D^{(A)}]$, before it reaches the defenders' "victory" state, $[A^{(D)}, D^{(D)}]$. The various possible definitions of $[A^{(A)}, D^{(A)}]$ and $[A^{(D)}, D^{(D)}]$ allow for a wide variety of objectives, including those that are explicitly asymmetric. For example, a naïve symmetric model based solely upon exhausting the enemy's forces would set $[A^{(A)}, D^{(A)}] = [A, 0]$ and $[A^{(D)}, D^{(D)}] = [0, D]$, for arbitrary positive values of $A$ and $D$, respectively; whereas a more sophisticated asymmetric model might set $[A^{(A)}, D^{(A)}] = [A, D^{*}]$ and $[A^{(D)}, D^{(D)}] = [0, D]$, for some $D^{*} > 0$ that is sufficiently humiliating for the defenders.

To compute the attackers' probability of victory directly from equations (7) and (8) is a formidable task. However, employing a method proposed by Powers (1995) for monitoring insurance-company solvency, we can transform the bivariate system of equations (7) and (8) into a univariate system by identifying a function $U = g(A, D)$ such that:

$$dU = \frac{\partial g(A,D)}{\partial A} dA + \frac{\partial g(A,D)}{\partial D} dD = -\left[\frac{\partial g(A,D)}{\partial A} \frac{k_1}{v^q} AD + \frac{\partial g(A,D)}{\partial D} k_2 A\right] dt = U dt.$$

Equations (7) and (8) then can be replaced with:

$$dU = U dt + \sigma dZ, \tag{9}$$

where $dZ$ is a standard Brownian motion and $\sigma = \sigma(U, t) > 0$ is the associated infinitesimal standard deviation[5].

After several standard manipulations, we find that:

$$U = g(A,D) = \exp\left(-\frac{(v^q/k_1)\sqrt{(2/k_2)}}{\sqrt{(v^q A/k_1) - (D^2/2k_2)}} \tan^{-1}\left(\sqrt{\frac{(D^2/2k_2)}{((v^q A/k_1) - (D^2/2k_2))}}\right)\right).$$

Setting $[A^{(A)}, D^{(A)}] = [A, 0]$ to identify the attackers' victory with total target destruction, and letting $\sigma = \sigma(U, t)$ be a positive constant, it then follows that:

$$p = \Pr\{\text{Target destruction}\} = \Pr\{\text{Attackers win}\} = \frac{\Phi((\sqrt{2}/\sigma)U(0)) - (1/2)}{\Phi(\sqrt{2}/\sigma) - (1/2)}, \tag{10}$$

where $U(0) = g(A(0), D(0))$ and $\Phi()$ is the cumulative distribution function of the standard normal distribution.

## Conclusions

A close examination of equation (10) yields the following results[6]:

- If:

$$D(0) \geq \sqrt{\frac{2v^q k_2 A(0)}{k_1}},$$

  then $p = 0$: for a terrorist attack of any fixed size, a sufficiently large defensive force can prevent target destruction with certainty.

- If:

$$D(0) < \sqrt{\frac{2v^q k_2 A(0)}{k_1}},$$

  then:

$$\frac{\partial p}{\partial D(0)} < 0, \quad \lim_{D(0) \to \sqrt{2v^q k_2 A(0)/k_1}} p = 0, \quad \text{and} \quad \lim_{D(0) \to 0} p = 1$$

  as the size of the initial defensive forces increases, the chance of target destruction decreases, and eventually is eliminated; as the size of the initial defensive forces shrinks to zero, target destruction is assured.

- If:

$$D(0) < \sqrt{\frac{2v^q k_2 A(0)}{k_1}},$$

  then:

$$\frac{\partial p}{\partial A(0)} > 0, \quad \lim_{A(0) \to \infty} p = 1, \quad \text{and} \quad \lim_{A(0) \to k_1 [D(0)]^2 / 2v^q k_2} p = 0,$$

  as the size of the initial terrorist forces increases, the chance of target destruction increases, eventually achieving certainty; as the size of the initial terrorist forces shrinks to its lower bound, the chance of target destruction is eliminated.

- If:

$$D(0) < \sqrt{\frac{2v^q k_2 A(0)}{k_1}},$$

  then:

$$\frac{\partial p}{\partial \sigma} < 0, \quad \lim_{\sigma \to \infty} p > 0, \quad \text{and} \quad \lim_{\sigma \to 0} p = 1,$$

  as the combat uncertainty of a terrorist attack increases, the chance of target destruction decreases, but cannot be eliminated; however, as the combat uncertainty shrinks to zero, target destruction is assured.

- If:

$$D(0) < \sqrt{\frac{2v^q k_2 A(0)}{k_1}},$$

then:

$$\frac{\partial p}{\partial v^q} > 0, \; \lim_{v^q \to \infty} p < 1, \; \text{and} \; \lim_{v^q \to k_1 [D(0)]^2 / 2k_2 A(0)} p = 0,$$

as the physical domain of a terrorist attack expands, the chance of target destruction increases, but cannot achieve certainty; however, as the physical domain shrinks to its lower bound, the chance of target destruction is eliminated.

## Notes

1. See the TRIA of 2002.
2. The Terrorism Risk and Insurance Extension Act (TRIEA) of 2005 extended most of TRIA's provisions an additional two years, and TRIEA of 2007 subsequently extended them a further seven years.
3. The author is grateful to Waleed Al Mannai and Ted Lewis for raising this issue.
4. This is similar to the formulation of Perla and Lehoczky (1977).
5. Note that equation (9) is not mathematically identical to the systems (7) and (8) because the stochastic element, $\sigma \mathrm{d}Z$, is not derived from $\sigma_1 \mathrm{d}Z_1$ and $\sigma_2 \mathrm{d}Z_2$. Nevertheless, the assumption of a Brownian motion in equation (9) is, *ceteris paribus*, just as valid as the Brownian-motion assumptions in equations (7) and (8).
6. Importantly, these results are borne out for alternative functional forms of the infinitesimal standard deviation; specifically, $\sigma(U, t) = \sigma\sqrt{U}$ and $\sigma(U, t) = \sigma U$.

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